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Emergence of scale-free network with chaotic units

Pulin Gong*, Cees van Leeuwen

Laboratory for Perceptual Dynamics, Brain Science Institute, Riken, 2-1, Hirosawa, Wako-Shi, Saitama 351-0198, Japan

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Abstract

To study the evolution of complex network with dynamical units, in this paper we consider the development of the network with chaotic units. By the addition of new nodes continuously and the adaptive rewiring of the connections according to the dynamic coherence of the activity patterns in the network, we can obtain that the growing network self-organizes into a complex network of which the connectivity distribution reveals a power law, at the same time, the network has a high clustering coefficient and small average shortest path length. The importance of chaos in the emergence of this type of scale-free network is investigated through comparing it to systems of periodic and stochastic units. The functional advantage of the self-organized network with dynamical units is revealed by showing the robustness of the spatiotemporal dynamics of the complex network.

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Keywords: Scale-free network; Chaos; Emergence; Dynamical clusters

1. Introduction

A distinctive feature of complex systems is the emergent order resulting from their many interacting elements. Emerging order in system behavior as well as in structure have been studied extensively. For biological systems, it is natural to assume that

* Corresponding author. Laboratory for Perceptual Dynamics, Brain Science Institute, Riken, 2-1, Hirosawa, Wako-Shi, Saitama 351-0198, Japan. Tel.: +81-48-462-1111x7435; fax: +81-48-467-7236.
E-mail address: plgong@brain.riken.go.jp (P. Gong).

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behavior shapes structure and structure conditions behavior. This principle is applied here to the study of network structures. Important hidden regularities have been found in network structure, following the study of small-world networks by Watts and Strogatz [1]. Small-world networks are featured by the combination of a high clustering coefficient and small characteristic path lengths.

Recently, Barabasi and Albert presented the scale-free network, of which the distribution $P(k)$ of vertex connectivity, the number of connections for each unit in the network, decays as a power law, that is, free of a characteristic scale [2]. Scale-free distribution of connectivity provides networks with robustness with respect to random perturbation or removal of a proportion of the network connections [3]. Robustness is a highly desirable characteristic for real networks. Accordingly, a diversity of systems such as the world-wide web, social networks, the metabolic networks of life-sustaining chemical reactions inside cells, and protein interaction networks all belong to the highly heterogeneous family of scale-free networks [4–7].

In most of these systems, the structure has emerged spontaneously, rather than by design. In this perspective, an important issue is how this could have happened. In the initial study, Watts and Strogatz created small-world networks by randomly rewiring a regular network with probability p. Starting from initially regular conditions may be a useful strategy for artificial system design, it is implausible however for natural system development and evolution. For the scale-free networks, Barabasi and Albert proposed a model based on two mechanisms: growth and preferential attachment [2]. Growth means the cumulative addition of new units; preferential attachment means that the probability of attachment of a newly created node is proportional to the connected degree of the target nodes, thus richly connected nodes tend to get ret richer. More recent models in addition allow nodes to age so they can no longer accept new links, or vary the form of preferential attachment in such a manner that a node acquires new links with other monotonically increasing function not limited to linear preferential attachment functions [8–10]. In Ref. [11], a model was presented to take into account the case that in real systems new nodes of growing network will process only information concerning a subset of existing nodes. All of these features purport to enhance the realism of network development.

Despite the realistic features added, none of these studies have used the resources of the dynamics as provided by network itself in order to explain the emergence of structure. However, in some real biological networks such as the neuronal, genetic, and metabolic network models, units with oscillatory activity are very common [12–14]. For this reason it may be of great interest to study if such scale-free networks could emerge, resulting from spatiotemporal activity in a growing network with dynamical units. Some systems are always organized in a hierarchical way. For example, in visual system, the visual cortex is connected hierarchically for different areas. For the scale-free network, the connections of the network is inhomogenous and hierarchical. So, the emergence of a scale-free network may help us, in principle, to understand the development of the visual system. Some studies have shown that developing neural circuits undergo a period of rewiring, through which some connections are eliminated while others are added and this rewiring of connectivity depends on the coherent electrical activity in the circuit [15]. This process of growth and adaptive rewiring according to the dynamical coherence is the subject of our study about the development of complex network with dynamical units.

2. The model about the evolution of the network with dynamic units

We present a model for growth combined with adaptive rewiring according to the oscillatory dynamics of the network units. Our simulation studies demonstrate that a scale-free network with a high clustering coefficient is obtained if the oscillatory activity is chaotic. Chaos is a common phenomenon in nature, and it has been proposed to account for adaptive information processing in brains [16–18]. In Ref. [19], the study showed that for the globally coupled chaotic map with variable connection weighs, the units spontaneously separate into two groups, with one group possessing especially many outwardly directed connections to the other group. In Ref. [20], the authors suggested a role for chaotic neurons in determining network properties, i.e. the topology of interconnections. Here, we go further and propose that complicated chaotic behaviors are also important for the formation of the common scale-free complex networks.

In this study, for the purpose of the computational convenience, we use the chaotic logistic map. The logistic map shown in Eq. (1) is one of a generic family of one-hump maps, which has widespread relevance as a prototype of chaos and could be considered as a highly simplified model of neuron firing and population dynamics [21,22]. Coupled logistic maps are described by Eq. (2),

$$x(n+1) = f(x(n)) = 1 - ax(n)^2 , \tag{1}$$

$$x_i(n+1) = (1-\varepsilon) f(x_i(n)) + \frac{\varepsilon}{M_i} \sum_{j} {}_{j \in B(i)} f(x_j(n)) , \tag{2}$$

where $x_i(n)$ is the activity of the ith $(1 \leqslant i \leqslant N)$ unit at the nth time step. Where N is the total number of units in the current network. In the present study the number N of the current network is increased by adding new nodes one by one. $B(i)$ denotes the set of all the neighbors of the unit i, and M_i is the number of the units in the current set $B(i)$. The neighbors of the unit i are those units that have direct connections with unit i. The connections are bi-directional, this means that if unit i is a neighbor of unit j, j is also a neighbor of i. ε is the coupling strength. Throughout the paper, the coupling strength is fixed to be $\varepsilon = 0.4$. The parameter a is the system parameter which controls the dynamics of the each unit. We define the coherence $d_{ij}(n)$ between unit i and unit j as the absolute value of the difference between the activation values of the units, as follows:

$$d_{ij}(n) = |x_i(n) - x_j(n)| . \tag{3}$$

We start from a sparsely, fully connected, small random network with the total number of units M_0 linked by a number of connections L_0. A model for growth combined with adaptive rewiring according to spatiotemporal dynamics of the network with dynamical units is described as follows:

(I) Add a new node i_n with m connections to m different nodes in the current network randomly.

(II) Choose random initial activation values in the range $(-1, 1)$ for all units of the new network. Calculate the state of the system according to the Eq. (2), and discard an initial transient time T.

(III) Then the dynamical coherence at time $T+1$, between the new added node i_n and all the other units in new current network $d_{i_n j}(T+1)$ is calculated. We obtain the unit $j = j_1$, for which the value $d_{i_n j}(T+1)$ is minimum amongst all the other units. Furthermore, we obtain the unit $j = j_2$ for which $d_{i_n j_2}(T+1)$ is maximum amongst the neighbors of new unit i_n.

(IV) If unit j_1 is one of the neighbors of the unit i_n, then no change in the connections is made. Otherwise the connection between units i_n and j_2 is replaced by a connection between units i_n and j_1.

(V) Go to step (II) and repeat the algorithm for K_0 times.

(VI) Go to step (I) and repeat adding a new node.

3. The evolution of the network

In the following study, $M_0 = 50$, $m = 14$, $L_0 = 850$, $K_0 = 75$, $T = 2000$. To obtain chaotic activity in the network units, in Eq. (2) we choose parameter $a = 1.796$. Note that our result does not change qualitatively by using other values within the chaotic range. After t iterations of the model, a complex network with $M_0 + t$ nodes results. Fig. 1 shows the distribution of its connections for $t = 1700$. The connectivity in the network reveals a heavy-tailed distribution. The network has evolved into a scale-free

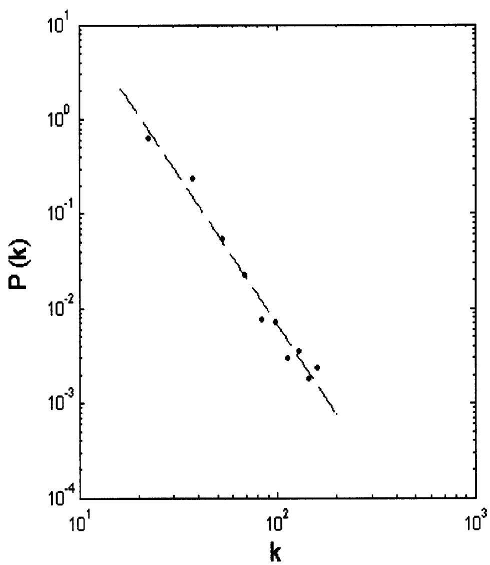

Fig. 1. (Log–log plot) Distribution of the connections of the self-organized network with chaotic units when $t = 1700$. The dashed line is obtained by the least-squares fit of the original data.

state with the probability a node has k connections, following a power-law $p(k) = k^{-\gamma}$, with the exponent $\gamma = 3.09 \pm 0.17$. The clustering coefficient and the average shortest path length of the final network are calculated according to Watts and Strogatz [1]. We obtain for our network the clustering coefficient $C_1 = 0.15$, and the average shortest path length $L_1 = 2.70$. It has been found that many real networks present a clustering coefficient much larger than the corresponding random graph. In order to compare with the corresponding random graph of the self-organized scale-free network, we generated a random graph by connecting nodes randomly, making sure that the numbers of nodes and connections between them are the same as those of the obtained self-organized network. For the random graph, clustering coefficient and shortest path length, respectively, are $C_0 = 1.7 \times 10^{-2}$, $D_0 = 2.56$. We observe that $C_1 \gg C_0$, and $D_1 \geqslant D_0$, that is, the clustering coefficient is much larger than the corresponding random network, and the shortest path length is close to that of the random network. We may, therefore, conclude that a growing network with chaotic units, according to our model produces a scale-free network with the characteristics of small-world network. We have also run our model by different values for m, k_0, and T, and obtained the similar results.

Preferential attachment of growing network is a very common feature of realistic growing networks [2,12]. We use the manner similar to that presented in Ref. [23] to study the preferential attachment in our model. For our model about the growing network with chaotic units, we consider the set of nodes at time t, and record the connections for all the nodes in the set. The connections for the set of nodes are denoted by K_i $(1 \leqslant i \leqslant M_0 + t)$. At time $t + \Delta t$, $\Delta t \ll t$, we measure the connections J_i of the set of nodes at time t. So we can obtain the increasing connections for the set of nodes at time t, which are $\Delta k_i = J_i - K_i$. In Fig. 2, we show the histogram of the average increasing attachment for the set of nodes at time t, the values for t and Δt are $t = 1650$, and $\Delta t = 250$. We can see that there is the preferential attachment for our model about growing network. In the earlier models preferential attachment is accomplished by attaching the new node to the target nodes according to their degree of connectivity [2,8–10]. Thus, the decision where to attach a node is based on detailed knowledge about the connectivity distribution of the entire network. It seems the new node knows the whole distribution of the connectivity. This is an unlikely requirement for real large-scale networks. By contrast in our model with dynamical units preferential attachment is an emergent property based on information about network dynamics. It might be considered more plausible for dynamical networks that this decision can arise from within the system itself.

In order to investigate the mechanisms responsible for the emergence of the scale-free network, we study some variants of our model. The first variant is one in which the dynamics of the units is periodic. For this purpose we choose in Eq. (2) the control parameter $a = 0.51$, making that the units are in a period-1 state. The second variant is one in which the dynamics of the single unit is stochastic. For this purpose we use the random generator to take the place of the logistic function. For the two variants, we start from the same small random network and run our model for the same number of times as for the chaotic model in the above study. We calculate the distributions of the connections for the two variants. The results are shown in Fig. 3. Clustering coefficients (C) and average shortest path lengths (L) of the two variants are shown in

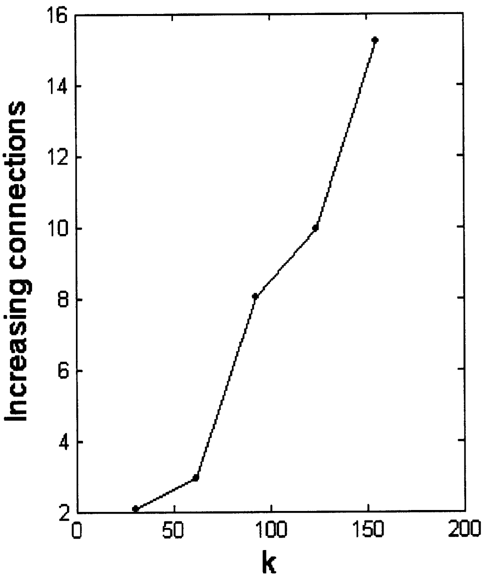

Fig. 2. The average increasing attachments versus the number of the connections at time t, where $t = 1650$, and $\Delta t = 250$.

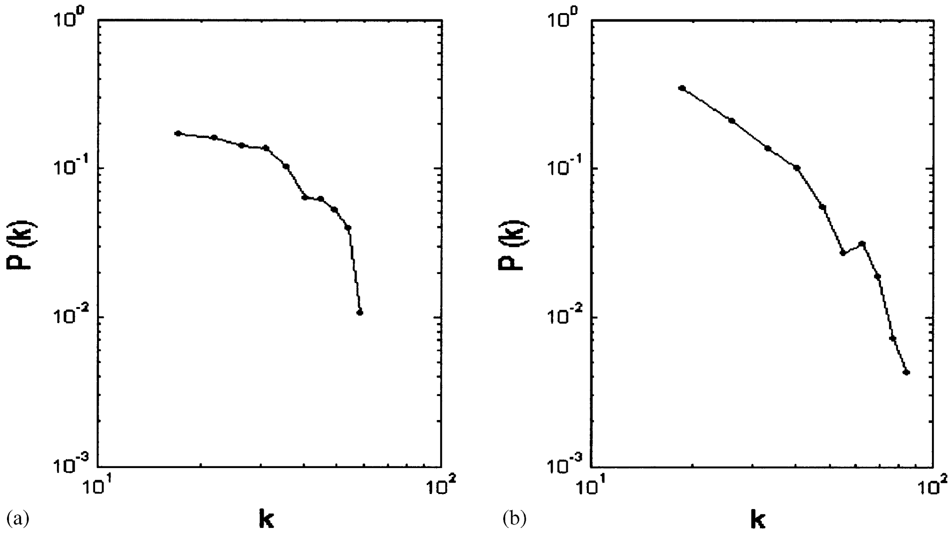

Fig. 3. (Log–log plot) The distribution of the connections for the self-organized network with different dynamical units: (a) periodic units when $t = 1700$; (b) stochastic units when $t = 1700$.

Table 1. For comparison, the clustering coefficient and average shortest path length of the corresponding random graph with the same units and connections are also shown in Table 1.

Table 1
Clustering coefficient (C) and average shortest path length (L) for some cases

	C	L
Periodic units	0.024	2.61
Stochastic units	0.025	2.69
Chaotic units	0.152	2.70
Random graph	0.017	2.56

For the periodic units, we observe that the distribution does not have a scale-free structure. Moreover, the clustering coefficient in Table 1 is shown to be still close to the corresponding random graph. The same is true for the stochastic units. For the stochastic units, the distribution of the connections, however, has at least a scale-free part, be it with a cutoff. The study of these two variants, therefore, indicates the unique importance of chaotic activity for the emergence of a scale-free network with small-world network characteristics.

4. Robust feature of spatiotemporal dynamics

Locally and globally coupled chaotic maps have been studied extensively and some interesting dynamics have been found [13,24]. For scale-free small-world networks, the connectivity is of a distinct kind, intermediate between locally and globally coupled, and between random and regular and this topology may add some distinctive characteristics to the dynamics of coupled chaotic maps. We investigate the dynamics of our scale-free network of coupled maps for its robustness. The robustness of the structure of scale-free network has been studied in terms of the change of average shortest path length under lesioning [3]. But for the purpose of studying the evolution of a network with dynamic units, the spatiotemporal dynamics characteristics of the networks are of central importance, because these are to provide the functional significance of the networks. For instance, the spatiotemporal oscillatory dynamics supporting by the neural circuits is very important for information binding and integration [25].

For a randomly coupled map lattice, there is no exact synchronization phase [26], but mutual fuzzy synchronization or fuzzy dynamical clusters are very common. Within the fuzzy dynamical clusters the units are coherent and almost synchronized. In a fuzzy dynamical cluster, for every pair of elements in the cluster i, j, their distances satisfy $|x_i(n) - x_j(n)| < \delta$. In our study, the value is chosen as $\delta = 0.05$. For the scale-free network obtained in the above section, after discarding a long time transient, the size of each dynamical cluster (the number of participating units) is calculated for every iteration. The distribution of the sizes of the dynamical clusters are calculated over a long time period (100,000). The result is shown in Fig. 4. We observe in Fig. 4 that the distribution has a power-law part followed by an exponential cut (black line). We then remove 6% of the nodes randomly, and obtained the distribution shown in Fig. 4, red line. We observe that the distribution is almost same to that of the original network.

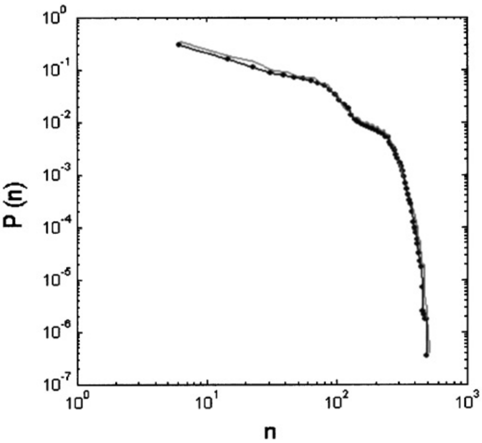

Fig. 4. (Log–log plot) The distribution of the sizes of the dynamical clusters in a very long time period for the self-organized scale-free network. The black line gives the distribution of the original self-organized network, and the red line gives the distribution of self-organized network when as many as 6% of the nodes are removed.

Thus, for the scale-free network with dynamical units, the spatiotemporal dynamics of the complex network is robust. The robustness of the saptiotemporal dynamics of the self-organized scale-free network is based on the robustness in the structure of the network [3]. In the scale-free network, a small number of nodes with large numbers of connections certainly play a very important role in the spatiotemporal dynamics. Random lesioning is unlikely to have much effect, as long as it does not involve these nodes.

5. Summary

In conclusion, we investigate the development of networks with chaotic units. By the addition of new nodes one by one and adaptive rewiring of the connections according to the dynamical coherence of the nodes, the network self-organizes into a scale-free network. At the same time the self-organized network has a high clustering coefficient and small characteristic path lengths. Moreover, the spatiotemporal dynamics of the network appear to be robust with respect to random lesioning.

In an earlier study, we presented model in which adaptive rewiring was performed on an initially random network without growth [27]. In this study, a small-world network was obtained which, however, did not have scale-free characteristics. Instead, the connectivity showed a Poisson distribution. From both studies it may be concluded that, whereas the chaotic dynamics of the units and the connections according to the unites' dynamic coherence are essential for obtaining small-world networks through self-organization, for scale-free networks in addition growth is needed. More detailed analysis about the spatiotemporal dynamics of the scale-free network in a broad parameter space is now under way and its results will be reported in due course.

The emergence of a scale-free network with dynamical units is based on chaotic dynamical behaviors, growth, and adaptive rewiring according to the dynamical coherence of the nodes. The model may be used to help us to understand the formation of the structured networks in some real systems with dynamical units. For example, the role of adaptive rewiring according to the spatiotemporal activities of neural circuits is very important for neural development [15]. On the other hand, it is well known that chaotic behaviors are very common in neural systems. Chaos has been observed in some spatial hierarchies of neural systems [28,29], and many different views have been given on the function of chaotic behavior. In the present study we show that chaos is essential for the emergence of a common scale-free network structure. We hope that our study can incur more interest in the role of chaotic behaviors in the formation of structures of some complex systems such as the brain.

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